Validating Hughe's Flow (Meeting 6)

So for today's work I have doen the following. Seeing the hughe's paper, I see that that the numerical method to solve the governing equation were not explained clearly, so I have use of another research paper, whcih also solves hughe's flow, with some better methods. but my objective was to see the method of solving the equation, and validating my results with it. Here is the link to the paper : Link

Problem Statement:

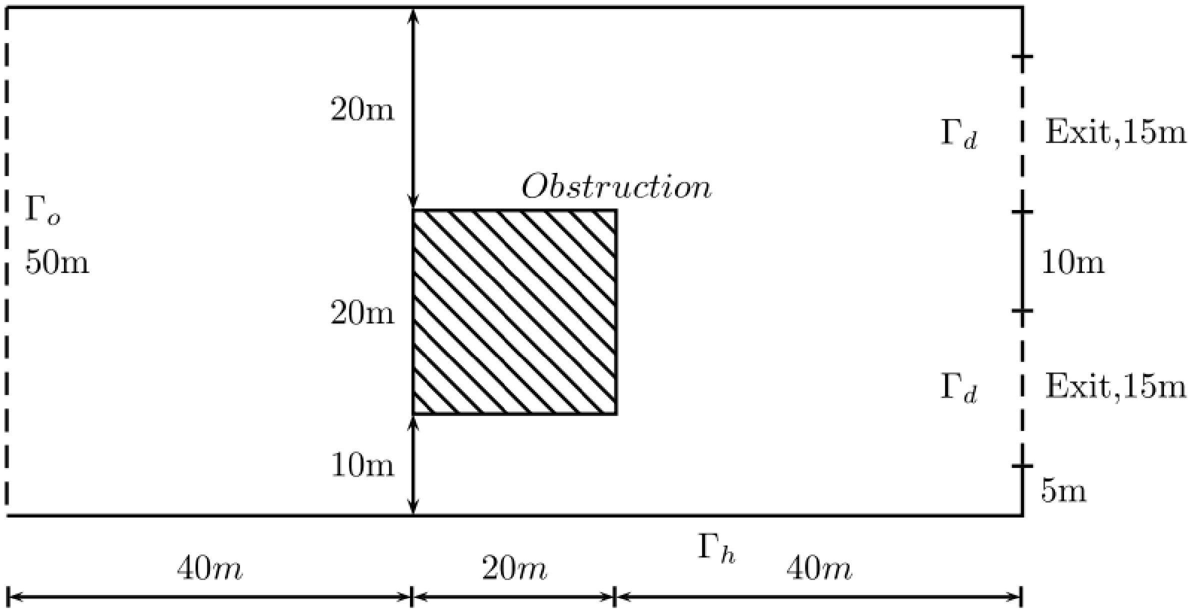


Fig. 1. The geometry of the railway platform.

From the paper: → Greenshield's model

$N_x = 100$, $L_x = 100$ M
$N_y = 50$ $L_y = 50$ M

$U(\rho) = 2\left(1 - \frac{\rho}{10}\right)$ → $V_{max} = 2$ M/s, $\rho_{max} = 10$ Ped/M²

$g(\rho) = (0.002\,\rho^2)$

$flux = (\rho \cdot \bar{u}) = \bar{f}$

$f_x = (\rho u)$

$f_y = (\rho v)$

Inflow condition:

on boundary:

$x = 0 ; 0 \le y \le 50$

$$f_1 = \begin{cases} (t/12) & ; 0 \le t \le 60 \\ (10 - t/12) & ; 60 \le t \le 120 \\ 0 & ; 120 \le t \le 300 \end{cases}$$

$f_2 = 0 \;\; \forall t$

$f_1 = (\rho \cdot u)$

$f_2 = (\rho \cdot v)$

At inflow since $v = 0$ therefore $u = a$

$f_1 = \rho\left(2\left(1 - \frac{\rho}{10}\right)\right)$

$\rho = 5 \pm (25 - 5 f_1)^{1/2}$

Inflow boundary condition:

$$\boxed{\rho(t) = 5 \pm (25 - 5 f_1(t))^{1/2}}$$

Obstacle/wall

$\frac{\partial \phi}{\partial n} = 0$

$\rho = 0$

$\phi_{obstacle} = 10^{12}$

Outflow condition

~~$\phi_i = \phi_{i-1}$~~ $\phi = 0$

~~$\rho_i = \rho_{i-1}\phi$~~ $\rho_i = \rho_{i-1}$

NOTE:

$g(\rho) = 0.002\,\rho^2$

Defining the Domain

```
clc;
clear all;
Nx = 100; Ny = 50;
Lx = 100; Ly = 50;
dx = Lx/Nx; dy = Ly/Ny;
[x, y] = meshgrid(linspace(dx,Lx,Nx), linspace(dy,Ly,Ny));
```

Setting the Parameters

```
Vmax = 2;
rho_max = 10;
CFL = 0.4;
```

```
obstacle = false(Ny, Nx);
obstacle(10:30,40:60) = true;
```

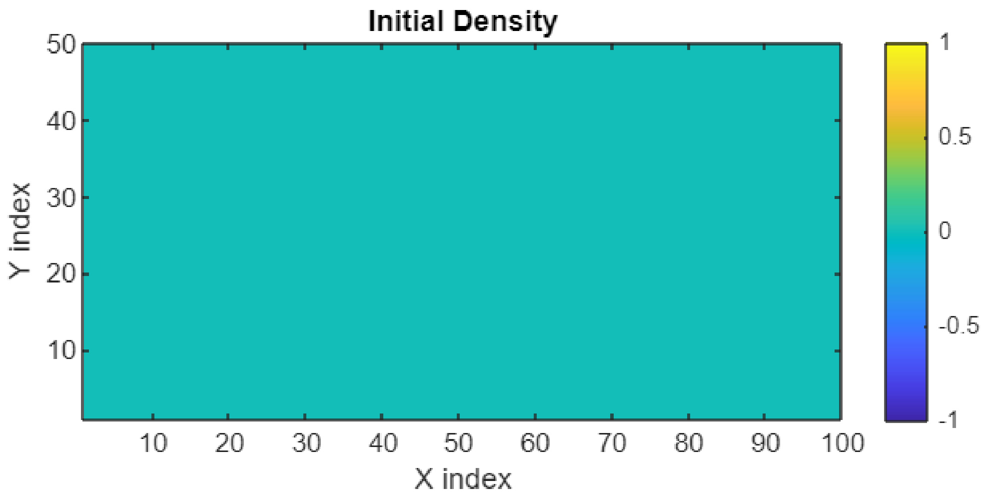
Initialising the Initial Condition

```
% Domain setup;
rho = 0 * ones(Ny, Nx);

% === Plot ===
figure;
contourf(rho, 20, 'LineColor', 'none');
```

```
Warning: Contour not rendered for constant ZData
```

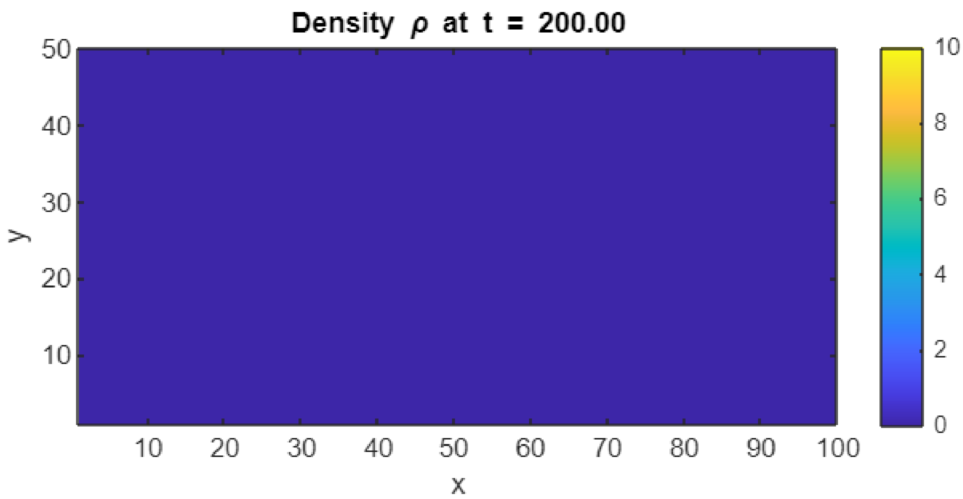
```
colorbar;
axis equal tight;
xlabel('X index');
ylabel('Y index');
title('Initial Density');
```

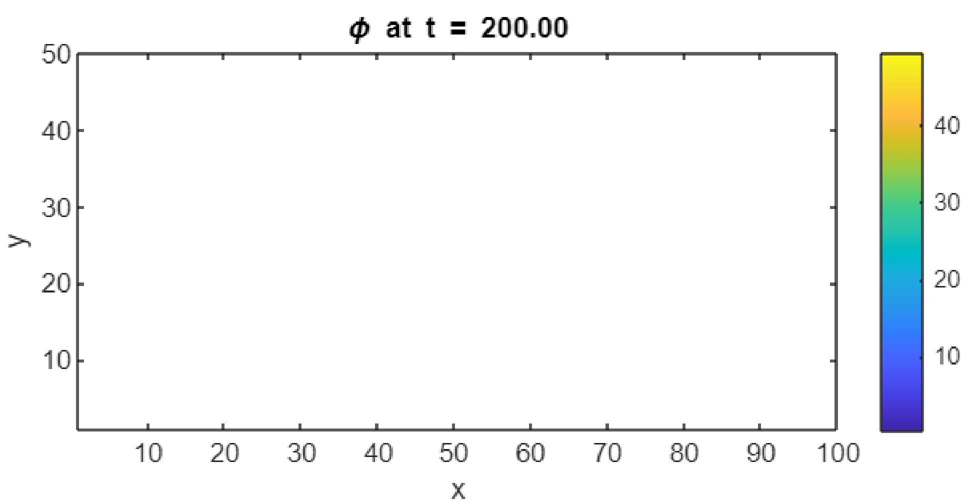


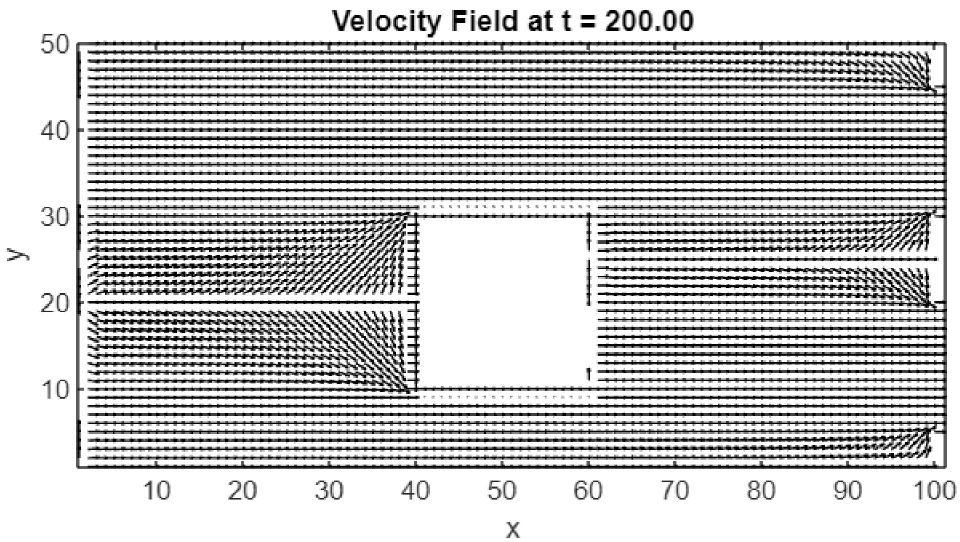
The main function for time marching

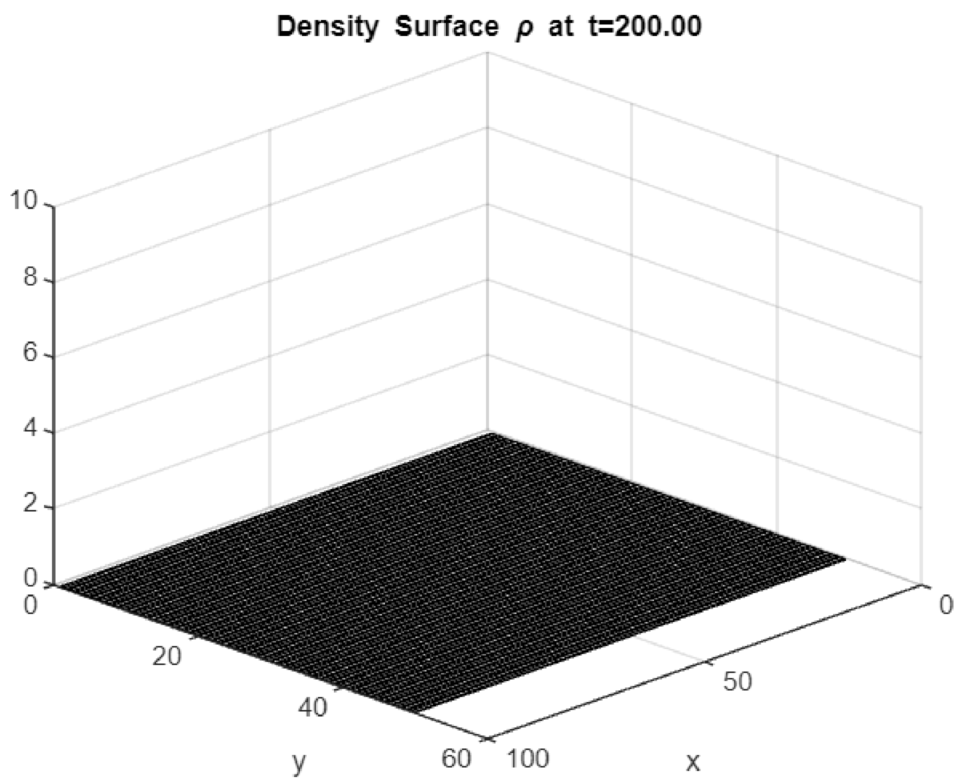
```
Tfinal = 200;
t = 0;
while t < Tfinal
    % Speed field
    f = Vmax*(1 - rho/rho_max);
    % To speed doesn't become zero completely
    f(f<1e-6) = 1e-6;
    % Solve eikonal equation
    phi = fast_sweeping(1./f,dy,dx,Nx,Ny,obstacle);
    % Direction field
    [phix, phiy] = gradient_phi(phi, dx, dy,Nx,Ny,obstacle);
    grad_mag = sqrt(phix.^2 + phiy.^2) + 1e-12;
    dirx = -phix ./ grad_mag;
    diry = -phiy ./ grad_mag;
    % Velocity field
    vx = f .* dirx;
    vy = f .* diry;
    % Time step
    maxspeed = max(max(sqrt(vx.^2 + vy.^2)));
    dt = CFL * min(dx,dy) / maxspeed;
    if t+dt > Tfinal, dt = Tfinal-t; end
    % Update rho
    rho = upwind_update(t,rho, vx, vy, dx, dy, dt,rho_max,obstacle);
```

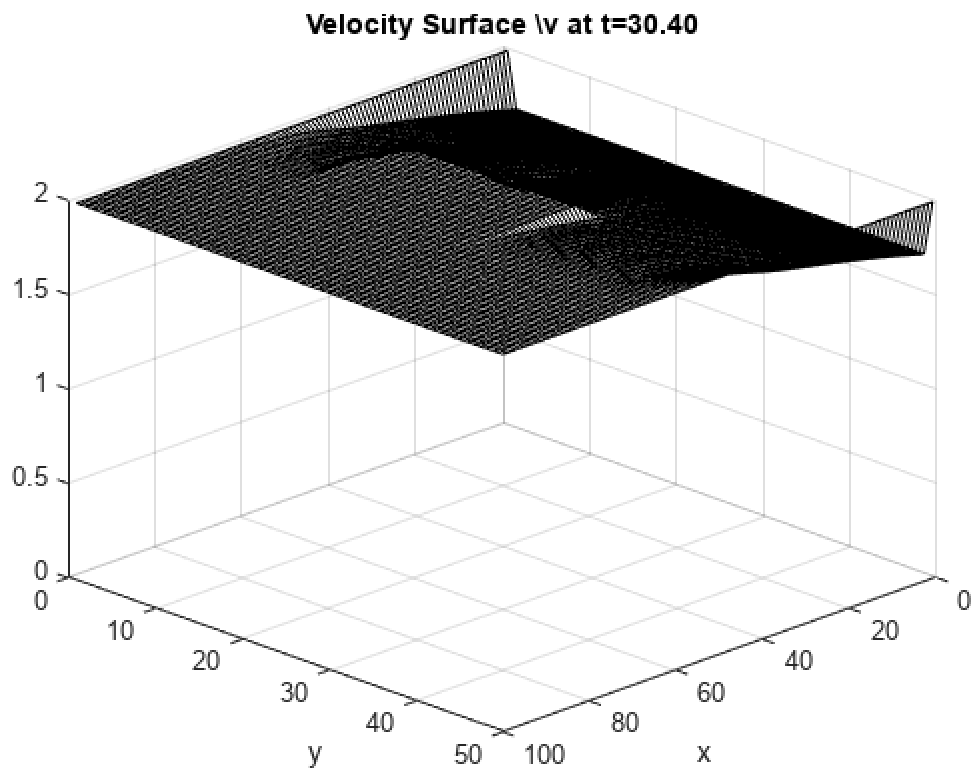
```
    % Advance time
    t = t + dt;
    % Plots
    if mod(round(t/dt),2) == 0
        % Density (phi)
        figure(1);
        contourf(x, y, rho, 20, 'LineColor', 'none');
        colorbar;
        caxis([0 rho_max]);
        title(sprintf('Density \\rho at t = %.2f', t));
        xlabel('x'); ylabel('y');
        axis equal tight;
        % Potential (phi)
        figure(2);
        contour(x, y, phi, 100, 'LineColor', 'none');
        colorbar; title(sprintf('\\phi at t = %.2f',t));
        xlabel('x'); ylabel('y');
        axis equal tight;
        % Velocity field
        figure(3);
        quiver(x, y, vx, vy, 1, 'k');
        title(sprintf('Velocity Field at t = %.2f',t));
        xlabel('x'); ylabel('y');
        axis equal tight;
        % Density Surface
        figure(4);
        surf(x, y, rho, 'EdgeColor', 'black', 'FaceColor', 'none');
        view(45+90, 30);
        title(sprintf("Density Surface \\rho at t=%.2f",t));
        xlabel('x'); ylabel('y');
        zlim([0 rho_max]);
        % Velocity Surface
        figure(5);
        surf(x, y, f, 'EdgeColor', 'black', 'FaceColor', 'none');
        view(45+90, 30);
        title(sprintf("Velocity Surface \\v at t=%.2f",t));
        xlabel('x'); ylabel('y');
        zlim([0 Vmax]);
        drawnow;
    end
end
```


Density ρ at t = 200.00
y
x
50
40
30
20
10
10
20
30
40
50
60
70
80
90
100
10
8
6
4
2
0


ϕ at t = 200.00
y
x
50
40
30
20
10
10
20
30
40
50
60
70
80
90
100
40
30
20
10


Velocity Field at t = 200.00
50
40
30
20
10
y
10
20
30
40
50
60
70
80
90
100
x


Density Surface ρ at t=200.00
10
8
6
4
2
0
0
20
40
y
60
100
50
0
x




```
%This function will calculate the flux at the right boundary when needed.
function flux = flux_value(t)
    if t>=0 && t<=60
        flux = t/12;
    elseif t>=60 && t<=120
        flux = 10 - (t/12);
    elseif t>=120
        flux = 0;
    end
end
```

```
function [phix,phiy] = gradient_phi(phi,dx,dy,Nx,Ny,obstacle)
    phix = zeros(Ny,Nx);
    phiy = zeros(Ny,Nx);
    % Top Boundary
    for i=2:Nx-1
        phix(Ny,i) = (phi(Ny,i+1) - phi(Ny,i-1))/(2*dx);
        phiy(Ny,i) = (phi(Ny,i) - phi(Ny-1,i))/dy;
    end
    % Bottom Boundary
    for i=2:Nx-1
        phix(1,i) = (phi(1,i+1) - phi(1,i-1))/(2*dx);
        phiy(1,i) = (phi(2,i) - phi(1,i))/dy;
    end
    % Left Boundary
```

```
    for j = 2:Ny-1
        phix(j,1) = (phi(j,2) - phi(j,1))/dx;
        phiy(j,1) = (phi(j+1,1) - phi(j-1,1))/(2*dy);
    end
    % Right Boundary
    for j = 2:Ny-1
        phix(j,Nx) = (phi(j,Nx) - phi(j,Nx-1))/dx;
        phiy(j,1) = (phi(j+1,Nx) - phi(j-1,Nx))/(2*dy);
    end
    % Interior point
    for i = 2:Nx-1
        for j = 2:Ny-1
            % If the interioir point is far from the obstacle
            if ~obstacle(j+1,i) && ~obstacle(j-1,i) && ~obstacle(j,i+1) &&
~obstacle(j,i-1)
                phix(j,i) = (phi(j,i+1) - phi(j,i-1))/(2*dx);
                phiy(j,i) = (phi(j+1,i) - phi(j-1,i))/(2*dy);
            % If the interior point is at the boundary of obstacle
            else
                if obstacle(j+1,i)
                    phiy(j,i) = (phi(j,i) - phi(j-1,i))/dy;
                elseif obstacle(j-1,i)
                    phiy(j,i) = (phi(j+1,i) - phi(j,i))/dy;
                end
                if obstacle(j,i+1)
                    phix(j,i) = (phi(j,i) - phi(j,i-1))/dx;
                elseif obstacle(j,i-1)
                    phix(j,i) = (phi(j,i+1) - phi(j,i))/dx;
                end
            end
        end
    end
    % Corners
    phix(1,1) = (phi(1,2) - phi(1,1))/dx;
    phiy(1,1) = (phi(2,1) - phi(1,1))/dy;

    phix(Ny,1) = (phi(Ny,2) - phi(Ny,1))/dx;
    phiy(Ny,1) = (phi(Ny,1) - phi(Ny-1,1))/dy;

    phix(Ny,Nx) = (phi(Ny,Nx) - phi(Ny,Nx-1))/dx;
    phiy(Ny,Nx) = (phi(Ny,Nx) - phi(Ny-1,Nx))/dy;

    phix(1,Nx) = (phi(1,Nx) - phi(1,Nx-1))/dx;
    phiy(1,Nx) = (phi(2,Nx) - phi(1,Nx))/dy;
end
```

Fast sweep Algorithm

```
function phi = fast_sweeping(invf,dy, dx, Nx, Ny,obstacle)
```

```
%--------------------------------------------
% Initialization
%--------------------------------------------
phi = inf(Ny, Nx);        % Initial potential field
phi(5:20, Nx) = 0;           % Exit boundary (e.g., right side)
phi(30:45,Nx) = 0;

% Set phi inside obstacle to Inf
phi(obstacle) = inf;

%--------------------------------------------
% Fast Sweeping Iterations
%--------------------------------------------
for sweep = 1:20
    % === Sweep 1: i→, j→
    for i = 2:Nx-1
        for j = 2:Ny-1
            if ~obstacle(j, i)
                phi = update_phi(phi, invf, i, j, dx);
            end
        end
    end

    % === Sweep 2: i←, j→
    for i = Nx-1:-1:2
        for j = 2:Ny-1
            if ~obstacle(j, i)
                phi = update_phi(phi, invf, i, j, dx);
            end
        end
    end

    % === Sweep 3: i←, j←
    for i = Nx-1:-1:2
        for j = Ny-1:-1:2
            if ~obstacle(j, i)
                phi = update_phi(phi, invf, i, j, dx);
            end
        end
    end

    % === Sweep 4: i→, j←
    for i = 2:Nx-1
        for j = Ny-1:-1:2
            if ~obstacle(j, i)
                phi = update_phi(phi, invf, i, j, dx);
            end
        end
    end
```

```
        % Normal gradient will also be so in the case of obstacle
        phi(10:30,40) = phi(10:30,39);
        phi(10:30,60) = phi(10:30,61);
        phi(10,40:60) = phi(9,40:60) ;
        phi(30,40:60) = phi(31,40:60);

        phi(:, 1)   = phi(:, 2);       % Left

        phi(1:5,Nx) = phi(1:5,Nx-1);
        phi(5:20, Nx) = 0;           % Exit boundary (e.g., right side)
        phi(21:29,Nx) = phi(21:29,Nx-1);
        phi(30:45,Nx) = 0;
        phi(46:50,Nx) = phi(46:50,Nx-1);

        phi(1, :)   = phi(2, :);       % Top
        phi(Ny, :)  = phi(Ny-1, :);    % Bottom
    end
end

% ===========================================================
% Subfunction: Update one grid point using Godunov scheme
% ===========================================================
function phi = update_phi(phi, invf, i, j, dx)
    a = min(phi(j, i-1), phi(j, i+1));   % x-direction neighbors
    b = min(phi(j-1, i), phi(j+1, i));   % y-direction neighbors
    f = invf(j, i);   % Local inverse speed

    if abs(a - b) >= f * dx
        phi(j, i) = min(a, b) + f * dx;
    else
        inside = 2 * (f * dx)^2 - (a - b)^2;
        if inside >= 0
            phi(j, i) = (a + b + sqrt(inside)) / 2;
        end
    end
end
```

Up-winding scheme

```
function rho_new = upwind_update(t,rho, vx, vy, dx, dy, dt, rho_max,obstacle)
    [Ny, Nx] = size(rho);

    % ---------------------------------------------
    % Boundary Conditions (Neumann/Dirichlet mix)
    % ---------------------------------------------
    for j = 1:Ny
        rho(j,1) = 5-((25-5*flux_value(t))^0.5); % Inflow wall (Dirichlet)
    end
```

```
    rho(5:20,Nx)     = rho(5:20,Nx-1);
    rho(30:45,Nx)    = rho(30:45,Nx-1);
    rho(1:4,Nx)      = 0;
    rho(21:29,Nx)    = 0;
    rho(46:50,Nx)    = 0;


    for i = 1:Nx
        rho(1,i) = 0;                      % Top wall
        rho(Ny,i) = 0;                     % Bottom wall
    end

    rho(obstacle) = 0;                      % No density in obstacle

    % ---------------------------------------------
    % Initialize flux arrays
    % ---------------------------------------------
    flux_xp = zeros(Ny, Nx);
    flux_xm = zeros(Ny, Nx);
    flux_yp = zeros(Ny, Nx);
    flux_ym = zeros(Ny, Nx);
    % Upwind Scheme

    for j = 2:Ny-1
        for i = 2:Nx-1
            if ~obstacle(j, i)
                % x-direction fluxes
                flux_xp(j,i) = max(vx(j,i),0)*rho(j,i)   +
min(vx(j,i),0)*rho(j,i+1);
                flux_xm(j,i) = max(vx(j,i),0)*rho(j,i-1) + min(vx(j,i),0)*rho(j,i);

                % y-direction fluxes
                flux_yp(j,i) = max(vy(j,i),0)*rho(j,i)   +
min(vy(j,i),0)*rho(j+1,i);
                flux_ym(j,i) = max(vy(j,i),0)*rho(j-1,i) + min(vy(j,i),0)*rho(j,i);
            end
        end
    end

    % ---------------------------------------------
    % Update density using continuity equation
    % ---------------------------------------------
    rho_new = rho - (dt/dx)*(flux_xp - flux_xm) - (dt/dy)*(flux_yp - flux_ym);

    % ---------------------------------------------
    % Enforce bounds and obstacle exclusion
    % ---------------------------------------------
    rho_new = max(0, min(rho_max, rho_new));  % Clamp between 0 and max
end
```

Result Comparison

Here is the comparison of the results from the paper, and my own. Given below velocity field comparison.

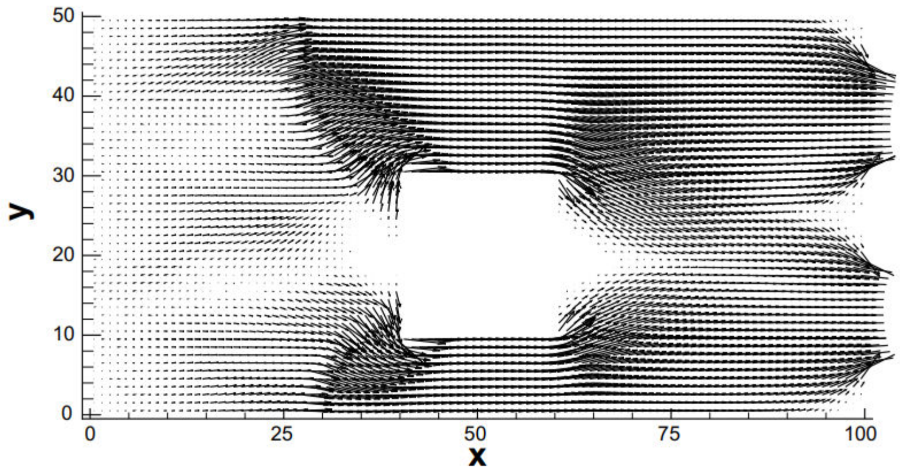
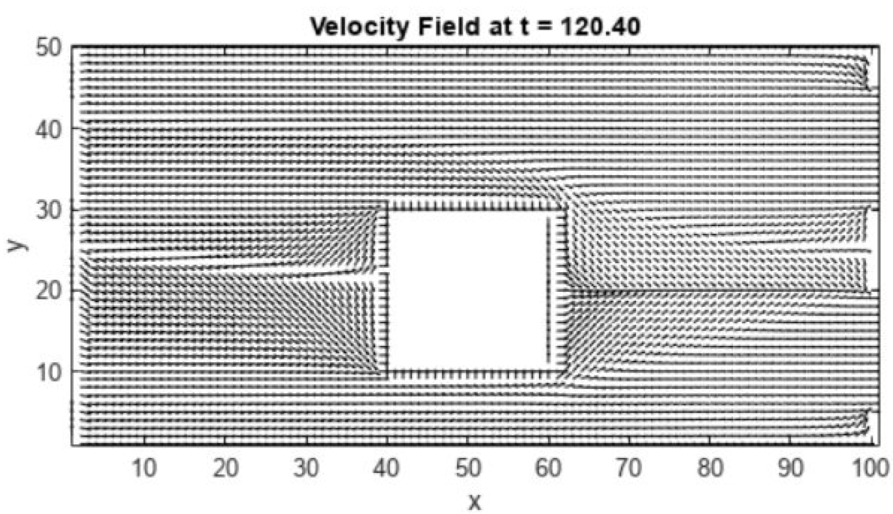
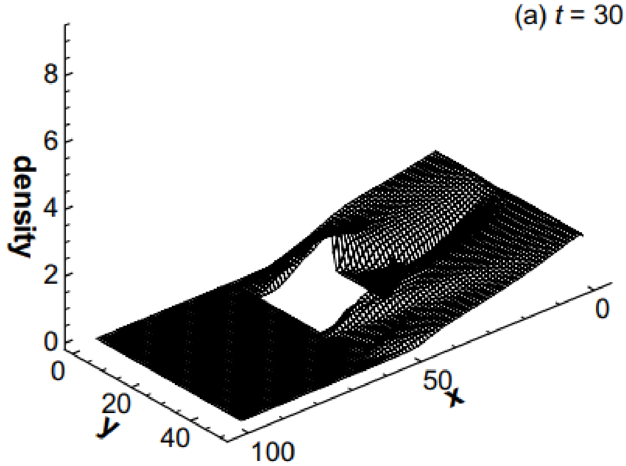


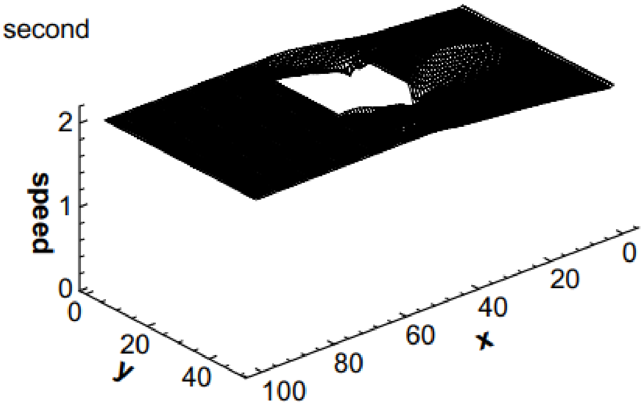
Fig. 3. The flow vector **f** at $t = 120$ s.

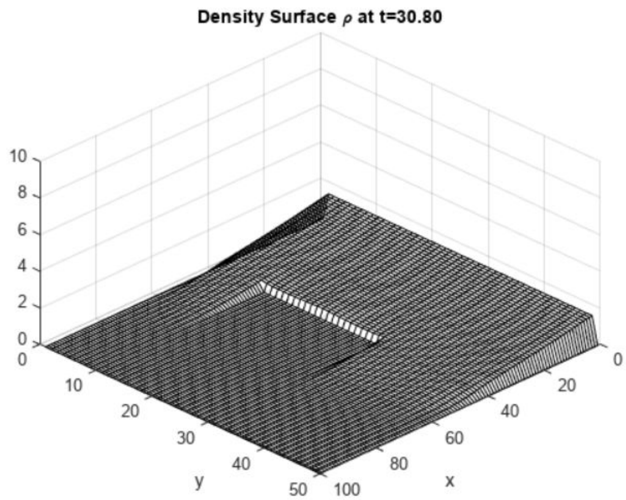


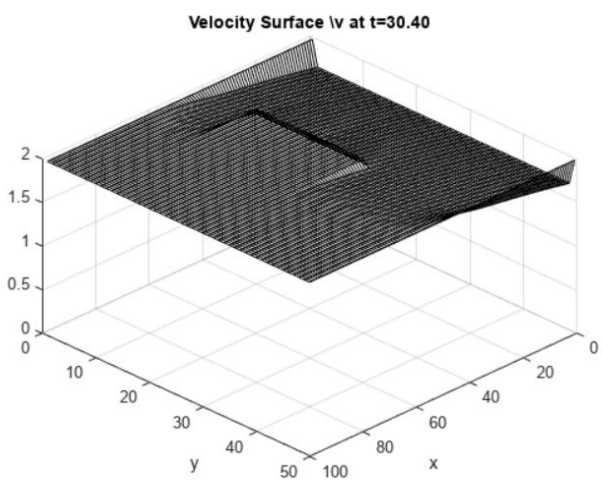
Density Comparison

(a) t = 30 second

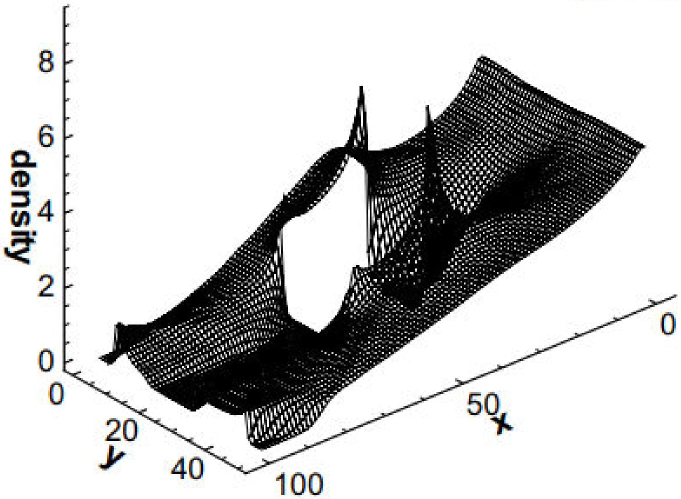

density
8
6
4
2
0
0
20
40
y
0
50
100
x

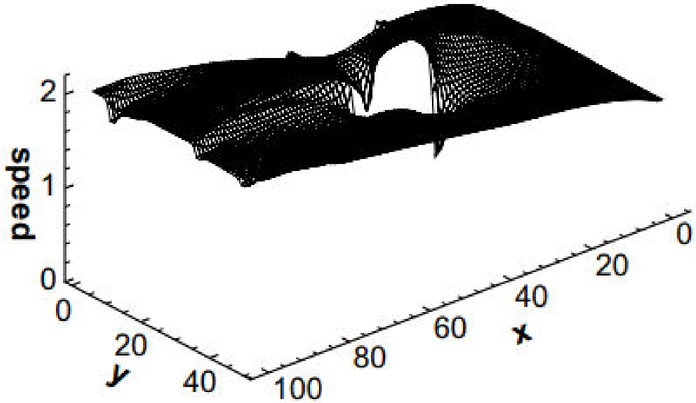

speed
2
1
0
0
20
40
y
0
20
40
60
80
100
x


Density Surface ρ at t=30.80
10
8
6
4
2
0
0
10
20
30
40
50
y
0
20
40
60
80
100
x

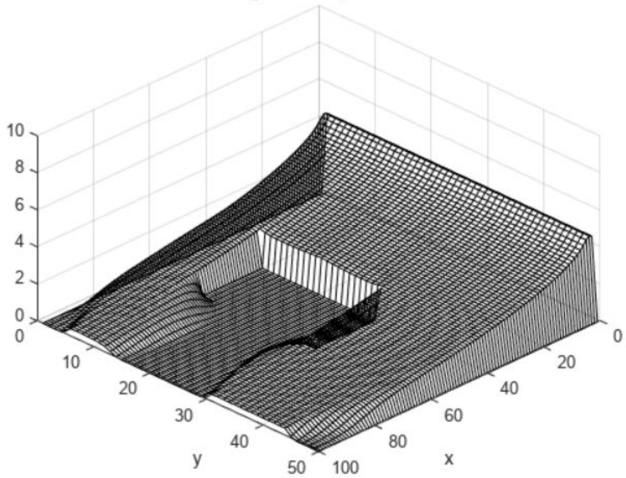

Velocity Surface \v at t=30.40
2
1.5
1
0.5
0
0
10
20
30
40
50
y
0
20
40
60
80
100
x

(b) t =60 second

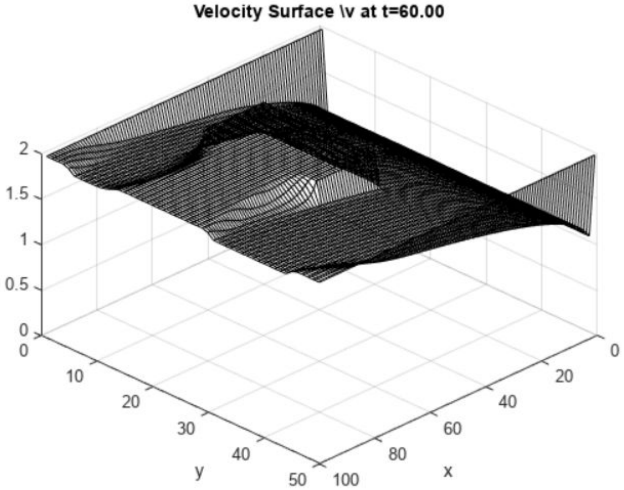




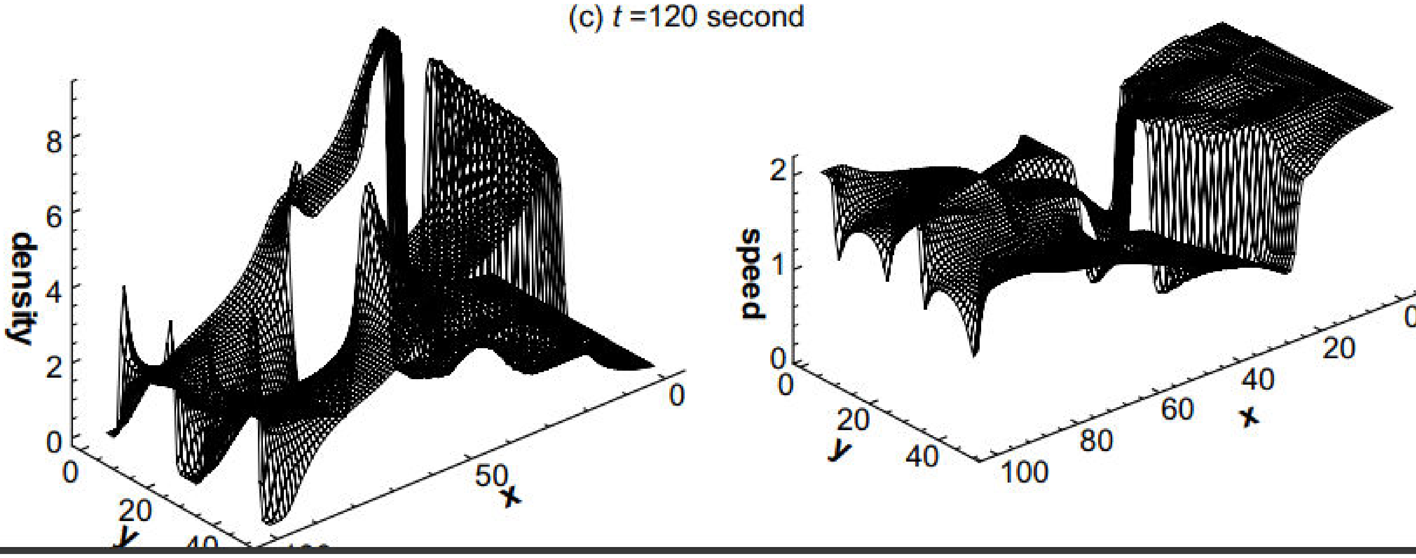
Density Surface ρ at t=60.80

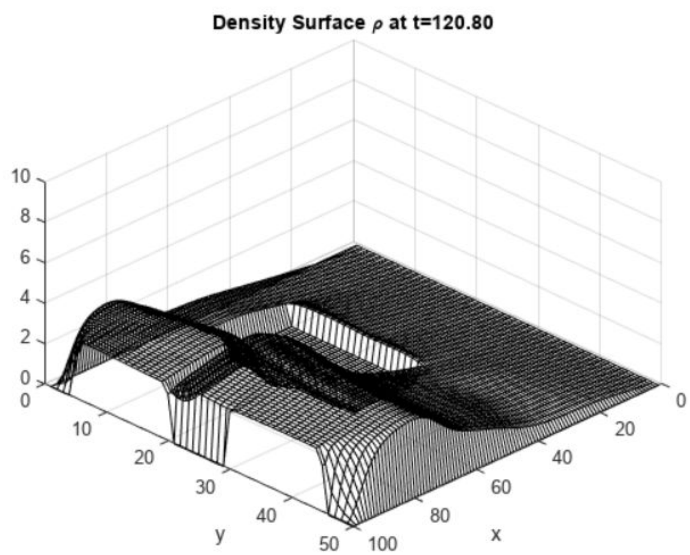


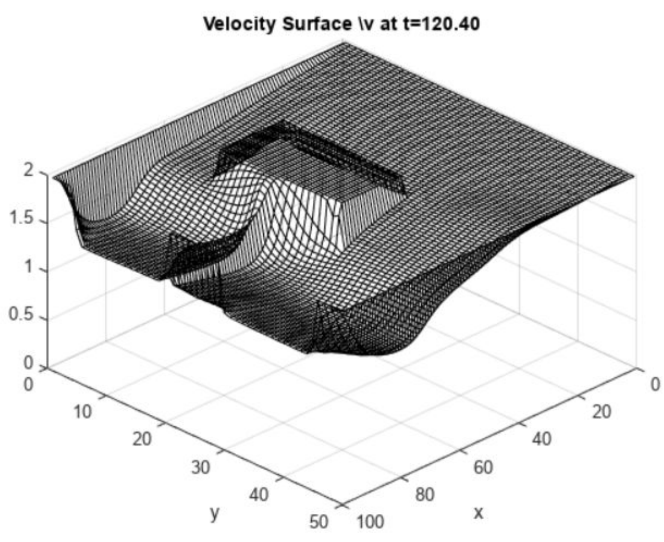
Velocity Surface \v at t=60.00



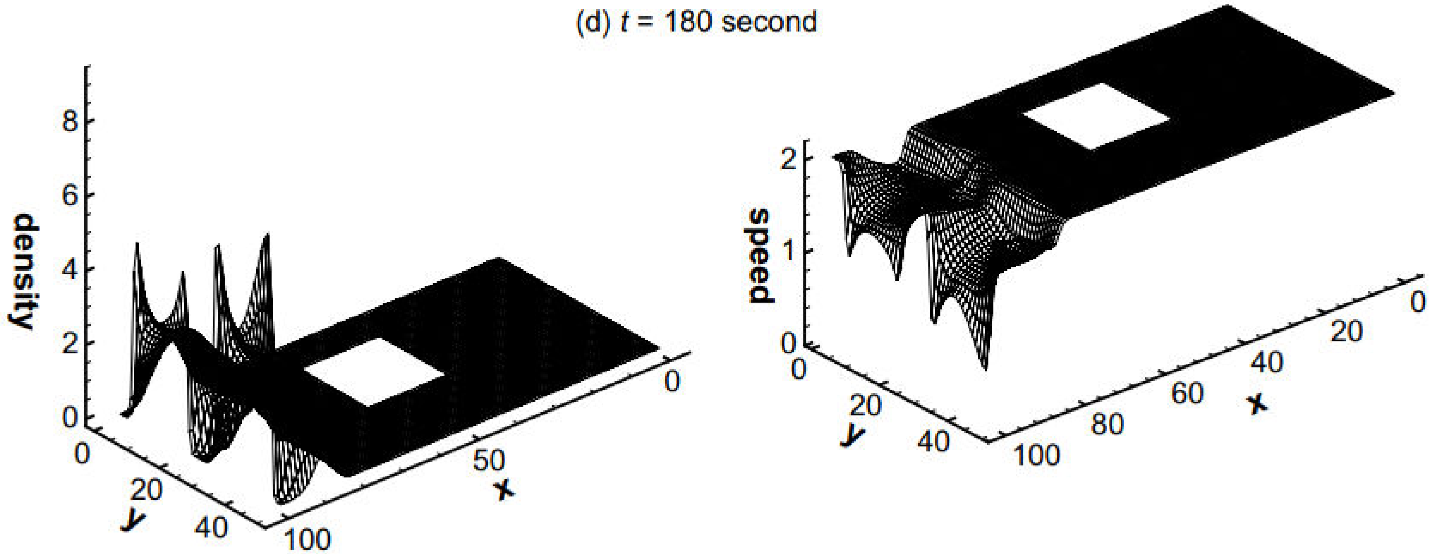
(c) t =120 second



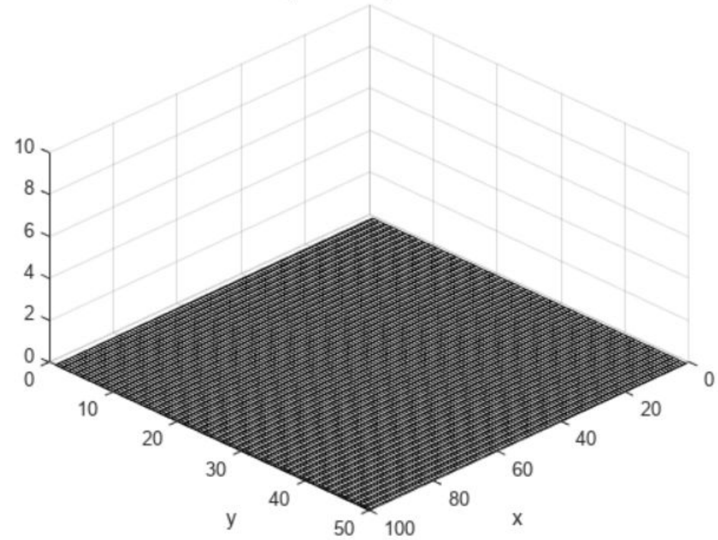




(d) t = 180 second



Density Surface ρ at t=180.80



Velocity Surface \v at t=180.00